MA 331 **Name (Print):** ____________________
Fall 2017
Final Exam Review

1.) Solve the following differential equations

a.) $xy' - 2y - x^3 \sin(x) = 0$
b.) $2y\frac{dy}{dx} = x^2 + 1$
c.) $\frac{dy}{dx} = \frac{e^x}{x^3} - \frac{3y}{x}$
d.) $\frac{dy}{dx} = \frac{3y}{x+1} + (x+1)^4$
e.) $\frac{dv}{dt} = 9.8 - 0.196v$

2.) Solve the following initial value problems

a.) $6y^2\frac{dy}{dx} = x(e^{x^2} + 2)$, $y(0) = 1$
b.) $\frac{dy}{dt} = \frac{4\sin(2t)}{y}$, $y(0) = 1$
c.) $\frac{dy}{dt} = -y\frac{1+2t^2}{t}$, $y(1) = 2$
d.) $ty' + 2y = t^2 - t + 1$, $y(1) = \frac{1}{2}$
e.) $\frac{dv}{dt} = 9.8 - 0.196v$, $v(0) = 48$

3.) Find the equilibrium points of the differential equation $y' = (y^2 - 4)(y+1)^2$. Determine the stability of the equilibrium points using a phase line.

4.) Find the general solution to the 2nd order differential equation

a.) $y'' + 6y' + 25y = 0$
b.) $y'' + 3y' - 10y = 0$
c.) $y'' - 2y' + y = 0$
d.) $y'' - 5y' = 0$
e.) $y'' - 12y + 36y = 0$
f.) $y'' - 4y' + 13y = 0$

5.) Solve the initial value problem

a.) $y'' - y' - 2y = 0, \quad y(0) = -2, \quad y'(0) = 5$
b.) $y'' + 2y' + 2y = 0, \quad y\left(\frac{\pi}{4}\right) = 2, \quad y'\left(\frac{\pi}{4}\right) = -2$
c.) $2y'' = -y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2$
d.) $y'' - y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 2$
e.) $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = 3$
f.) $y'' - 10y' + 29 = 0, \quad y(0) = 1, \quad y'(0) = 3$

6.) Consider a mass-spring system, given by $x''(t) + \gamma x'(t) + x = 0$. For what value of γ will the system be critically damped? What is the solution for the mass-spring system that is critically damped. You must show all work.

7.) For the following, compute the approximate solution using Euler's method. Recall that Euler's method is given by $t_{n+1} = t_n + h$ and $y_{n+1} = y_n + h * f(t_n, y_n)$ for the differential equation $y' = f(t, y)$. h represents the step size. You do not need to fill in the table.
$y' = 2t^2 + y$, $y(0) = 1$. Let h=0.5. Find Euler's approximation for $y(2)$

t_i	y_i	$f(t_i, y_i) =$	y_{i+1}

8.) Consider the following matrices:

$$\text{A} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}, \quad \text{B} = \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix},$$

Find the following quantities:

a.) AB b.) BA c.) $|A|$ d.) $|B|$
e.) A^{-1} f,) B^{-1}
g.) Eigenvalues and Eigenvectors of B

9.) Find the general solution to the system of equations

a.) $$\begin{aligned} x_1'(t) &= x_1 + x_2 \\ x_2'(t) &= 4x_1 + x_2 \end{aligned}$$

b.) $$\begin{aligned} x_1'(t) &= 2x_1 + 3x_2 \\ x_2'(t) &= 4x_1 + 3x_2 \end{aligned}$$

c.) $$\begin{aligned} x_1'(t) &= x_1 + 2x_2 \\ x_2'(t) &= -2x_1 + x_2 \end{aligned}$$

d.) $$\begin{aligned} x_1'(t) &= 6x_1 - 13x_2 \\ x_2'(t) &= x_1 \end{aligned}$$

e.) $$\begin{aligned} x_1'(t) &= 5x_1 + x_2 \\ x_2'(t) &= -4x_1 + x_2 \end{aligned}$$

f.) $$\begin{aligned} x_1'(t) &= -2x_1 + x_2 \\ x_2'(t) &= -x_1 \end{aligned}$$

10.) Solve the initial value problems

$$\text{a.)}\quad \begin{aligned} x_1'(t) &= 6x_1 - 3x_2, & x_1(0) &= -10 \\ x_2'(t) &= 2x_1 + x_2, & x_2(0) &= -6 \end{aligned}$$

$$\text{b.)}\quad \begin{aligned} x_1'(t) &= 3x_1 - 2x_2, & x_1(0) &= 1 \\ x_2'(t) &= 2x_1 - 2x_2, & x_2(0) &= -1 \end{aligned}$$

$$\text{c.)}\quad \begin{aligned} x_1'(t) &= -x_1 - 6x_2, & x_1(0) &= 0 \\ x_2'(t) &= -3x_1 + 5x_2, & x_2(0) &= 2 \end{aligned}$$

$$\text{d.)}\quad \begin{aligned} x_1'(t) &= -\frac{1}{10}x_1 + x_2, & x_1(0) &= 2 \\ x_2'(t) &= -x_1 - \frac{1}{10}x_2, & x_2(0) &= 2 \end{aligned}$$

$$\text{e.)}\quad \begin{aligned} x_1'(t) &= x_1 - 4x_2, & x_1(0) &= -2 \\ x_2'(t) &= 4x_1 - 7x_2, & x_2(0) &= 1 \end{aligned}$$

$$\text{f.)}\quad \begin{aligned} x_1'(t) &= -x_1 + x_2, & x_1(0) &= 1 \\ x_2'(t) &= -x_2, & x_2(0) &= 3 \end{aligned}$$

11.) Determine the stability of the (0,0) equilibrium from problems 9a-d.

12.) Determine the stability of the (0,0) equilibrium from problems 10a-d.

13.) Determine the type and stability of the equilibrium point (0,0) for the following systems.

$$\text{a.)} \begin{pmatrix} 2 & 7 \\ -5 & -10 \end{pmatrix}, \quad \text{b.)} \begin{pmatrix} -3 & 6 \\ -3 & 3 \end{pmatrix}, \quad \text{c.)} \begin{pmatrix} 6 & 8 \\ 2 & 6 \end{pmatrix}, \quad \text{d.)} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \text{e.)} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

14.) For what value(s) of a will the system below have a spiral point at (0,0)?

$$\bar{x}' = \begin{pmatrix} 5 & a \\ -2 & -1 \end{pmatrix} \bar{x}$$

15.) For the following nonlinear systems, find the equilibrium point(s) and determine the stability and type of each equilibrium point.

a)

$$\begin{aligned} x' &= xy + 3y \\ y' &= xy - 3x \end{aligned}$$

b)

$$\begin{aligned} x' &= xy^2 - 3xy + 2x \\ y' &= x + y - 1 \end{aligned}$$

c)

$$\begin{aligned} x' &= x^2 + y^2 - 13 \\ y' &= xy - 2x - 2y + 4 \end{aligned}$$

d)

$$
\begin{aligned}
x' &= 2 - x^2 + y^2 \\
y' &= x^2 - y^2
\end{aligned}
$$

e)

$$
\begin{aligned}
x' &= x^2y + 3xy - 10y \\
y' &= xy - 4x
\end{aligned}
$$

16.) Consider an alteration of the Lotka-Volterra model of predator-prey interactions. In this modification, we assume that the prey grow logistically (instead of exponentially). $x(t)$ denotes the population of rabbits (in thousands) while $y(t)$ denotes the population of foxes (in thousands). The equations are thus:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - bxy \\
\frac{dy}{dt} &= cxy - dy
\end{aligned}
$$

a) What are the assumptions in this model?

b) What do the parameters represent?

c) Find the equilibrium points and describe the physical meaning. Are there any conditions needed to ensure all equilibrium points are physically relevant (i.e., nonnegative)?

d) Assume $cK - d < 0$. Determine the stability and type of the biologically relevant equilibrium points.

e) What eventually happens to the rabbit and fox populations?

f) Assume $cK - d > 0$. Determine the stability and type of the equilibrium points.

g) What eventually happens to the rabbit and fox populations?

h) In order to increase the ending population of rabbits, what would need to change?

i) In order to increase the ending population of foxes, what would need to change?

17.) Consider the following model of infection spread through a population. Let $S(t)$ denote the susceptible population at time t and $I(t)$ denote the infectious population at time t.

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta SI + \gamma I \\
\frac{dI}{dt} &= \beta SI - \gamma I
\end{aligned}
$$

a) What are the assumptions in this model?

b) Show that $S(t) + I(t)$ is a constant number.

c) Using a), rewrite the system of 2 differential equations into a single differential equation (of I).

d) Find all equilibrium points

e) Assume that the parameters are such that the endemic equilibrium is physically relevant. Determine the stability of the equilibrium points using the phase line.

f) Assume that the parameters are such that the endemic equilibrium is not physically relevant. Determine the stability of the biologically relevant equilibrium point using the phase line.

g) What is the value of R_0?

18.) Consider the following simplified model of political opinion dynamics. Let $x(t)$ denote the fraction of people who are leftists, $y(t)$ the fraction of people who are rightists, and $z(t)$ the fraction of centrists. Assume there is no population growth and that everyone is either a leftist, rightist, or centrist. Leftists and rightists argue too much to change each other's minds, but leftists and rightists can be persuaded by centrists. Assume that political opinions can be changed at a rate proportional to the rate at which extremists and centrists meet. The equations describing these dynamics are given by:

$$\frac{dx}{dt} = rxz$$
$$\frac{dy}{dt} = ryz$$
$$\frac{dz}{dt} = -rxz - ryz$$

a) The parameter r can be positive or negative. What does the parameter r represent and what does it mean if it is positive or negative?

b) Show that $x + y + z = 1$

c) Using the fact that $z = 1 - x - y$, rewrite the system of 3 differential equations into a system of 2 differential equations.

d) Show that the solutions are given by $x = \frac{x_0 e^{rt}}{1+(x_0+y_0)(e^{vt}-1)}$, $y = \frac{y_0 e^{rt}}{1+(x_0+y_0)(e^{vt}-1)}$.

e) Assume $r > 0$. Find the limit of $x(t)$ and $y(t)$ as $t \to \infty$. What does this behavior represent in political terms?

f) Now Assume that $r < 0$.Find the limit of $x(t)$ and $y(t)$ as $t \to \infty$. What does this behavior represent in political terms?

19.) Consider the following *competition* model that describes the dynamics of populations of rabbits and sheep competing with each other for a common resource such as grass. Let $x(t)$ denote the population of rabbits (in thousands) at year t, and $y(t)$ denote the population of sheet (in thousands) at year t. The system of equations describing their populations is given by:

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x + ay}{K_1}\right)$$
$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y + bx}{K_2}\right)$$

a) What are the units for the parameters? What do you think the physical meanings are of the parameters r_1, r_2, K_1 and K_2?

b) What are the units for a and b? What do you think these represent?

c) Let K_1 =3, $r_1 = 3$, $a = 2$, $K_2 = 2$, $r_2 = 2$, and $b = 1$. Show that the equations can be written as:

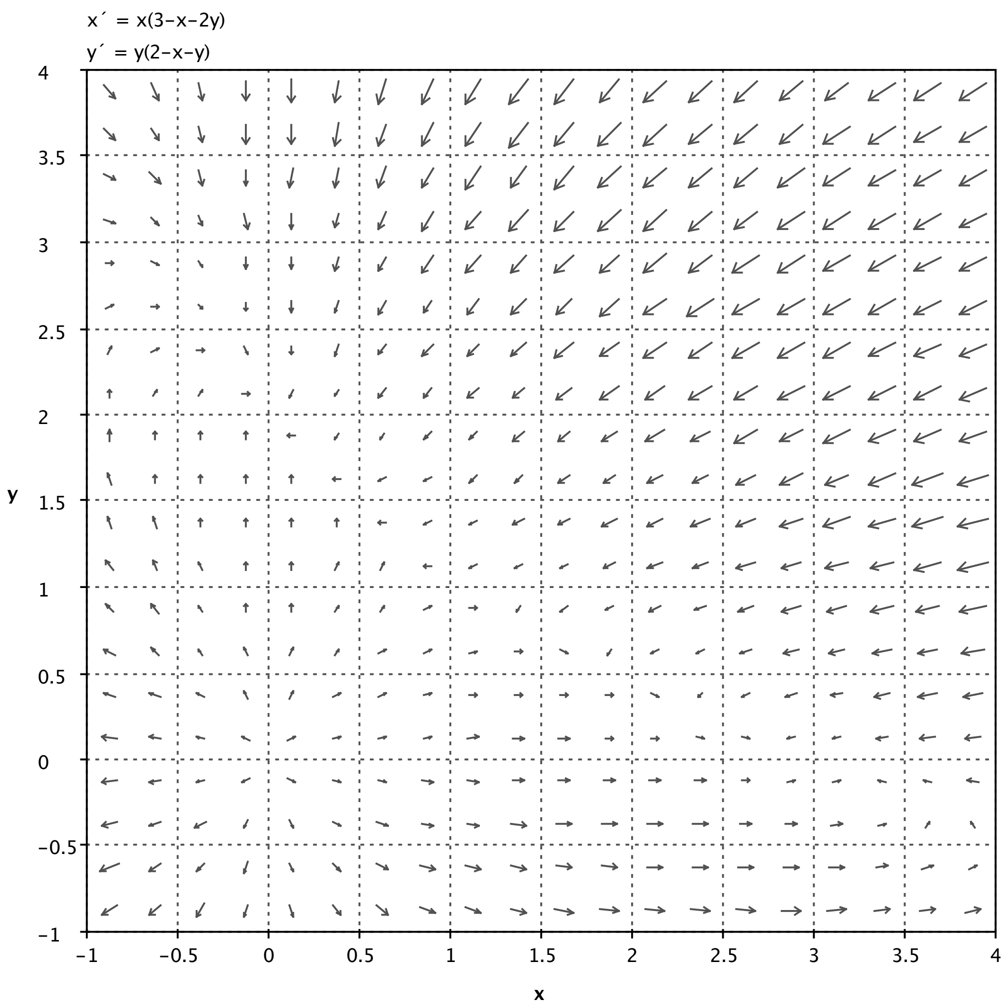
$$\frac{dx}{dt} = x\,(3 - x - 2y)$$
$$\frac{dy}{dt} = y\,(2 - x - y)$$

d) Find all equilibrium points of the equation given in c).

e) Determine the type and stability of all equilibrium points found in d).

f) What is the eventual outcome of the rabbits and sheep?

g) The phase plane for the competition model is below. Find an circle the equilibrium points on the phase plane. Draw the solution for the following initial conditions and state the long-term behavior, in biological terms.
- (2,1)
- (1,2)
- (3,2)



20.) The Figure below contains the direction field in the phase plane with trajectories. The initial conditions for the various trajectories are (x, y) = (1, 0.5), (0.5, 1), (4, 3), and (5, 2.5). The critical points are (0, 0), (0, 5), (7, 0), (3, 2). For each critical point, use the figure to determine the type of critical point: asymptotically stable node, unstable node, asymptotically stable spiral, unstable spiral, center (stable), saddle (unstable). Corroborate your findings by determining the equilibria and their stabilities analytically.