MA 331
Fall 2017
Final Exam Review Answers

1.) a.) $y(x) = Cx^2 - x^2\cos(x)$

b.) $y^2 = \frac{1}{3}x^3 + x + C$

c.) $y = \frac{e^x + C}{x^3}$

d.) $y = (x+1)^3\left(\frac{1}{2}x^2 + x + C\right)$

e.) $v = 50 + Ce^{-0.196t}$

2.) a.) $4y^3 = e^{x^2} + 2x^2 + 3$

b.) $y(t) = \sqrt{5 - 4\cos(2t)}$

c.) $y(t) = \frac{2}{t}e^{1-t^2}$

d.) $y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$

e.) $v = 50 - 2e^{-0.196t}$

3.) Equilibrium solutions occur at $y = -2$ (stable), $y = 2$ (unstable), and $y = -1$ (semi-stable).

4.) a.) $y = c_1e^{-3t}\cos 4t + c_2e^{-3t}\sin 4t$

b.) $y = c_1e^{2t} + c_2e^{-5t}$

c.) $y = c_1e^t + c_2te^t$

d.) $y = c_1e^{5t} + c_2$

e.) $y = c_1e^{6t} + c_2te^{6t}$

f.) $y = c_1e^{2t}\cos 3t + c_2e^{2t}\sin 3t$

5.) a.) $y = e^{2t} - 3e^{-t}$

b.) $y = \sqrt{2}e^{\frac{\pi}{4}}e^{-t}(\cos t + \sin t)$

c.) $y = 2e^{\frac{t}{2}} - e^{-t}$

d.) $y = \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}$

e.) $y = 2e^{-5t} + 13te^{-5t}$

f.) $y = e^{5t}(\cos(2t) - \sin(2t))$

6.) $\gamma = 2$. $x(t) = c_1e^{-x} + c_2xe^{-x}$

7.) 9.375

8.) a.)

$$\begin{pmatrix} 0 & 5 \\ 14 & 4 \end{pmatrix}$$

b.)

$$\begin{pmatrix} 5 & 13 \\ 5 & -1 \end{pmatrix}$$

c.) 7

d.) -10

e.)

$$\frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$$

f.)

$$-\frac{1}{10}\begin{pmatrix} -1 & -3 \\ -4 & -2 \end{pmatrix}$$

g.) $\lambda_1 = -5$, $v_1 = (1, -1)$, $\lambda_2 = 2$, $v_2 = (3, 4)$

9.) a.)

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} \quad (1)$$

b.)

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{6t} \quad (2)$$

c.)

$$\vec{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} \quad (3)$$

d.)

$$\vec{x} = c_1 e^{3t} \begin{pmatrix} 3\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 3\sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix} \quad (4)$$

e.)

$$\vec{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right] \quad (5)$$

f.)

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad (6)$$

10.) a.)

$$\vec{x} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + 4 \begin{pmatrix} -3 \\ -2 \end{pmatrix} e^{4t} \quad (7)$$

b.)

$$\vec{x} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \quad (8)$$

c.)

$$\vec{x} = e^{2t}\begin{pmatrix} -4\sin(3t) \\ 2\cos(3t)+2\sin(3t) \end{pmatrix} \tag{9}$$

d.)

$$\vec{x} = 2e^{-\frac{t}{10}}\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} \tag{10}$$

e.)

$$\vec{x} = e^{-3t}\begin{pmatrix} -12t-2 \\ 1-12t \end{pmatrix} \tag{11}$$

f.)

$$\vec{x} = e^{-t}\begin{pmatrix} 3t+1 \\ 3 \end{pmatrix} \tag{12}$$

11.) a. Saddle point, b) saddle point, c) unstable spiral, d) unstable spiral

12.) a. unstable node, b) saddle point, c) unstable spiral, d) stable spiral

13.)
a) Stable node
b) center
c) unstable node
d) stable spiral
e) saddle point

14.) $a < -\frac{9}{2}$

15.)
a) (0,0) center, (-2,2) saddle
b) (0,1) saddle, (1/4,3/4) unstable spiral
c) (2,3) saddle, (2,-3) saddle, (3,2) unstable node, (-3,2) stable node
d) (1,1) stable spiral, (1,-1) saddle, (-1,1) saddle, (-1,-1) unstable spiral
e) (0,0) saddle, (2,4) unstable node, (-5,4) stable node

16.)
a) prey grows logistically and dies at a rate proportional to the rate at which predator and prey meet. Predators grow in number when they eat prey (modeled at a rate proportional to the rate at which predator and prey meet) and, in the absence of prey, die at an exponential rate.
b) r is the intrinsic growth rate of the prey in the absence of predator. K is the carrying capacity in the absense of the predator. b represents the predation rate (fraction of prey population eaten per predator), c is a conversion rate of eaten prey into new predator populations and d is the per capita mortality rate.
c) (0,0) extinction of both species, $(K,0)$ survival of prey, $\left(\frac{d}{c}, \frac{r}{b}\left(1-\frac{d}{cK}\right)\right)$ coexistence. This last point requires that $cK > d$

d) (0,0) is a saddle point. $(K, 0)$ is stable.

e) Eventually the predator will die out and the prey reaches carrying capacity.

f) (0,0) is still a saddle point, $(K, 0)$ is now a saddle point, $\left(\frac{d}{c}, \frac{r}{b}\left(1 - \frac{d}{cK}\right)\right)$ is stable.

g) Predator and prey values stability to coexistence with prey population at $\frac{d}{c}$ and predator population at $\frac{r}{b}\left(1 - \frac{d}{cK}\right)$.

h) in order to increase final rabbit populations, we need the death rate of predators to be large (d large) or we need the growth of predator from eating prey to be small (c small).

i) . in order to increase final fox populations, we need faster growth rate of prey (r large), smaller rate at which prey are caught (b small), larger death rate of predators (d large), growth of predator from eating prey to be small (c small), or the carrying capacity for prey to be less (K small).

17.) a) Suscepitbles are infected at a rate proportional to the rate at which they meet infectious people. Infectious individuals remain infections for an average of $\frac{1}{\gamma}$ days, after which they become susceptible again. There is no immunity gained from contracting the disease.

b)

c)

$$\frac{dI}{dt} = \beta I(N - I) - \gamma I$$

d) $I^* = 0$, and $I^* = N - \frac{\gamma}{\beta}$.

e) (0,0) is unstable, $N - \frac{\gamma}{\beta}$ is stable.

f) (0,0) is stable

g) $R_0 = N\frac{\beta}{\gamma}$

h)

18.) a) The parameter r represents the persuasion abilities of centrists versus extremists. If $r > 0$, an extremist is more likely to convert a centrist. If $r < 0$, a centrist is more likely to convert an extremist.

b) Note that $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$, meaning that $x + y + z = C$, some constant C. Our assumptions imply that these are fractions and that everyone is either leftist, rightist or centrist, so $C = 1$.

c)

$$\frac{dx}{dt} = rx(1 - x - y)$$
$$\frac{dy}{dt} = ry(1 - x - y)$$

d)

e) $\lim_{t\to\infty} x(t) = \frac{x_0}{x_0+y_0}$, $\lim_{t\to\infty} y(t) = \frac{y_0}{x_0+y_0}$, $\lim_{t\to\infty} z(t) = 0$. Everyone becomes an extremist. The final fractions of the population who are rightist and leftist depend on how many were initially rightist or leftist.

f) $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} z(t) = 1$. Everyone becomes a centrist.

19.) a) Units of r_1, r_2 are /year, representing the intrinsic growth rate of the rabbits and sheep, respectively. K_1 and K_2 are in units of thousands and represent the individual carrying capacity of rabbits and sheep (when the other is not present).

b) a and b have no units. These represent the ability for one species to outcompete the other.

c)

d) (0,0), (0,2), (3,0), (1,1)

e) (0,0) is unstable node, (0,2) is stable node, (3,0) is a stable node), (1,1) is a saddle point.

f) They outcompete each other. Eventually only one species survives

g) (2,1) goes to all rabbits (sheep are extinct), (1,2) goes to all sheep (rabbits extinct), (3,2) goes to all rabbits (sheep extinct)

20.) (0,0) is unstable node, (3,2) is a saddle, (0,5) is a stable node, and (7,0) is a stable node.