Nonlinear Systems

Recall:

For a linear system of DEs $\vec{x}' = A\vec{x}$

(1) If $\det(A) \neq 0$, then $(0,0)$ is the only equilibrium point.

(2) The <u>stability</u> of the equilibrium point depends on the sign of evals λ_1, λ_2 (or $Re(\lambda_1), Re(\lambda_2)$)

- If $\lambda_1, \lambda_2 < 0$ it is stable
- If $\lambda_1, \lambda_2 > 0$ it is unstable
- If $\lambda_1 < 0 < \lambda_2$ it is saddle.

(3) The type of point the equilibrium is depends on the type of eigenvalue

- If λ_1, λ_2 are real & distinct, it is a node
- If λ_1, λ_2 are complex it is a spiral (if $Re(\lambda) \neq 0$) or a center (if $Re(\lambda) = 0$).

<u>Note</u>: We can also characterize equilibrium points <u>without</u> finding the eigenvalues.

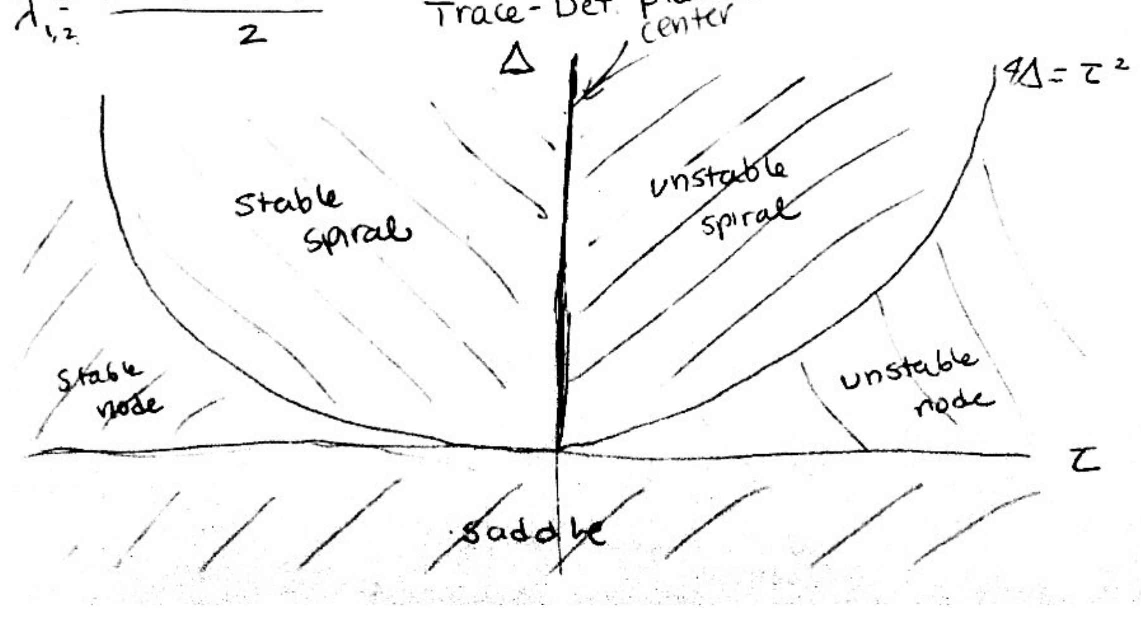
ex To find evals of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\tau}\lambda + \underbrace{(ad-bc)}_{\Delta} = 0$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Trace-Det plane



ex Use the Trace-Determinant Plane to classify the equilibrium (0,0) for the linear system of eqns

$$x' = 2x + 2y$$
$$y' = x + 3y$$

$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ $\tau = 5 > 0$ $\tau^2 - 4\Delta = 25 - 16 = 9$

$\Delta = 6 - 2 = 4 > 0$

$\Rightarrow$ unstable node.

ex Use the Trace-Determinant plane to classify (0,0) for the system

$$x' = x + y$$
$$y' = 4x + y$$

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ $\tau = 2 > 0$

$\Delta = 1 - 3 < 0$ $\Rightarrow$ saddle point.

What about nonlinear systems?

Consider the system

$$x' = f(x,y)$$
$$y' = g(x,y)$$

and suppose that (x^*, y^*) is an equilibrium pt (i.e. $f(x^*, y^*) = 0 = g(x^*, y^*)$)

Then we let $u = x - x^*$, $v = y - y^*$ (i.e., recenter & linearize abt fixed pt.)

Now let's see what happens

$$u' = x'$$
$$= f(x^* + u, y^* + v)$$
$$= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \underbrace{O(u^2, v^2, uv)}_{\text{Taylor series}}$$
$$= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv)$$

Similarly

$$v' = u\frac{\partial g}{\partial x}\bigg|_{(x^*,y^*)} + v\frac{\partial g}{\partial y}\bigg|_{(x^*,y^*)} + \underbrace{O(u^2, v^2, uv)}_{\text{ignore b/c small.}}$$

This becomes now

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

evaluate @ x^*, y^*

matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

is called the Jacobian matrix of the equilibrium point (x^*, y^*).

The system $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} \begin{pmatrix} u \\ v \end{pmatrix}$ can now be analyzed as a linear system (since A is a constant matrix).

ex
Find & classify all fixed pts of the system

$$x' = x^3 - x$$
$$y' = -2y.$$

(1) Find equilibrium pts

$0 = x^3 - x \Rightarrow x(x^2-1) = 0 \Rightarrow x^* = 0, \pm 1$

$0 = -2y \Rightarrow y^* = 0$

$(0,0)$, $(-1,0)$, $(1,0)$ are our eq. pts

(2) Find the Jacobian matrix

$$J = \begin{pmatrix} 3x^2-1 & 0 \\ 0 & -2 \end{pmatrix}$$

(3) Evaluate Jacobian for your fixed points to determine stability

- $(0,0)$

$J_{00} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\begin{matrix} \tau = -3 < 0 \\ \Delta = 2 > 0 \end{matrix}$ > stable $\begin{matrix} \tau^2 - 4\Delta \\ 9 - 8 > 0 \end{matrix}$ node

$(0,0)$ is a stable node

- $(-1,0)$

$J_{(-1,0)} = \begin{pmatrix} 3-1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\begin{matrix} \tau = 0 \\ \Delta = -4 < 0 \end{matrix} \Rightarrow$ saddle

$(-1,0)$ is a saddle point

- $(1,0)$

$J_{(1,0)} = J_{(-1,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$(1,0)$ is a saddle point.

Check on pplane.

ex Predator - Prey Problem.

Let $x(t)$ denote number of prey (rabbits) at time t.

$y(t)$ denote number of predator (fox) at time t.

An equation to describe their dynamics is given by

$$x' = \alpha x - \beta xy$$
$$y' = \delta xy - \gamma y.$$

What are the assumptions

Prey

1. The population of prey in the absence of predator grows proportional to the current population of prey with a constant of proportionality γ.
2. Prey is preyed upon at a rate proportional to the rate at which predators & prey meet. Note if either predators or prey are 0, there is no predation.

Predators

1. In the absence of prey ($x=0$), the predator population would decay to 0 exponentially (i.e., predators starve w/out food access)
2. The predator grows when it eats the prey (which is based on the meeting rate of pred & prey).

ex

Find fixed points & stability for

$$x' = 8x - 2xy$$
$$y' = 4xy - 8y.$$

① Find equilibria

$$0 = 8x - 2xy \Rightarrow 0 = 2x(4 - y) \Rightarrow x^* = 0 \text{ or } y^* = 4$$
$$0 = 4xy - 8y \Rightarrow 0 = 4y(x - 2) \Rightarrow y^* = 0 \text{ or } x^* = 2$$

if $x^* = 0 \Rightarrow y^* = 0$ $\quad (0,0)$

$y^* = 0 \Rightarrow x^* = 0$

$(2, 4)$

Find Jacobian

$$\begin{pmatrix} 8-2y & -2x \\ 4y & 4x-8 \end{pmatrix}$$

(3) Evaluate Jacobian at equilibrium pts

(0,0)

$$J_{(0,0)} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix} \Rightarrow \text{saddle.}$$

(2,4)

$$J_{(2,4)} = \begin{pmatrix} 0 & -8 \\ 8 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \tau = 0 \\ \Delta = 64 > 0 \end{array}$$

$0 - 32 < 0$.
$\Rightarrow$ center!
(since $\tau = 0$).

Examine phase portrait.

What improvements can be made to make this system more realistic?

Assume logistic growth for prey

$$x' = rx(1-x) - bxy$$
$$y' = cxy - dy$$