Epidemiological Models

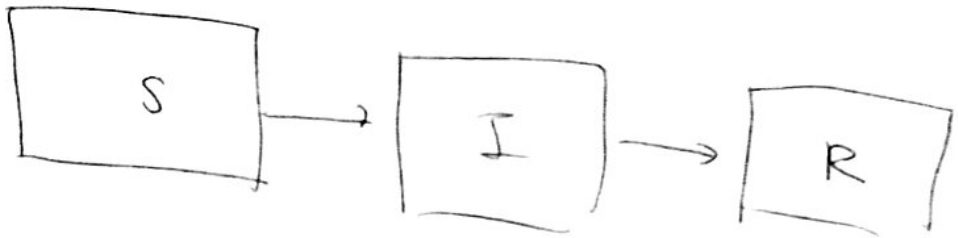
We can also explain the spread of disease through a population.

ex Influenza ~~chicken pox~~ spread through a dorm (campus)

Let $S(t)$ denote # of susceptible people at the t (thousands)

$I(t)$ denote # of infectious people at the t (thousands)

$R(t)$ denote # of recovered individuals at the t. (immune for a short amount of time). (thousands)



Assumptions

- No students drop out or enroll over the time period
- disease is transmitted at a rate proportional to the rate of encounters of susceptible ~~& infectious~~ & infected individuals (βSI)
- The average duration of infection is ~~m~~ days.
- Average duration of immunity is ~~n~~ days.

$$\frac{dS}{dt} = -\beta SI + \frac{1}{n}R$$

$$\frac{dI}{dt} = \beta SI - \frac{1}{m}I$$

$$\frac{dR}{dt} = \frac{1}{m}I - \frac{1}{n}R$$

Call $\gamma = \frac{1}{n}$, $\nu = \frac{1}{m}$

$$\frac{dS}{dt} = -\beta SI + \gamma R$$

$$\frac{dI}{dt} = \beta SI - \nu I$$

$$\frac{dR}{dt} = \nu I - \gamma R$$

NOTE $S+I+R=N$ a constant population

$R = N - S - I$

$$\frac{dS}{dt} = -\beta SI + \gamma(N - S - I)$$

$$\frac{dI}{dt} = \beta SI - \nu I$$

what are the possible steady states & stability?

① Eq $\quad 0 = -\beta SI + \gamma N - \gamma S - \gamma I$

$0 = \beta SI - \nu I$

↳ $0 = I(\beta S - \nu) \Rightarrow I^* = 0$ or $S^* = \nu/\beta$

if $I^* = 0$

$0 = \gamma N - \gamma S \Rightarrow S^* = N$

if $S^* = \nu/\beta$

$0 = -\beta(\nu/\beta) I^* + \gamma N - \gamma\nu/\beta - \gamma I^*$

$\nu I^* + \gamma I^* = \gamma N - \gamma\nu/\beta$

$I^*[\beta(\gamma + \nu)] = \gamma(N - \nu)$

$I^* = \frac{\gamma(N - \nu)}{\beta(\gamma + \nu)}$

Equilibrium points

$(N, 0)$ ← corresponds to disease free steady state (no epidemic)

$\left(\nu/\beta, \frac{N - (\nu/\beta)}{\nu + \gamma}\right)$ ← exists only if $N > \nu/\beta$ (since negate population doesn't make sense

Jacobian Matrix

$$J = \begin{pmatrix} -\beta I^* - \gamma & -\beta S^* - \gamma \\ \beta I^* & \beta S^* - \nu \end{pmatrix}$$

(3) Evaluate Jacobian at steady states to determine stability

$(N, 0)$

$$J_{(N,0)} \begin{pmatrix} -\gamma & -\beta N - \gamma \\ 0 & \beta N - \nu \end{pmatrix}$$

eigenvalues are on the diagonal

$-\gamma < 0$

$\beta N - \nu$ - depends on parameters

if $\beta N - \nu < 0$ $(N, 0)$ is stable

if $N < \nu/\beta$

i.e., there is no epidemic if β is small (transmission rate is low) or ν is large (i.e. avg duration of infection is short).

$\frac{N\beta}{\nu} >$ opposite R

$\left(\nu/\beta, \frac{N-(\nu/\beta)}{\nu+\gamma}\right)$ exists only if $N > \nu/\beta$

$$J_{\left(\nu/\beta, \frac{N-(\nu/\beta)}{\nu+\gamma}\right)} = \begin{pmatrix} -\beta\left(\frac{N-\nu/\beta}{\nu+\gamma}\right) - \gamma & -\nu - \gamma \\ \beta\left(\frac{N-\nu/\beta}{\nu+\gamma}\right) & \nu - \nu \end{pmatrix}$$

$$= \begin{pmatrix} -\left(\frac{\beta N - \nu}{\nu+\gamma}\right) - \gamma & -\nu - \gamma \\ \frac{\beta N - \nu}{\nu+\gamma} & 0 \end{pmatrix}$$

$\tau = -\left(\frac{\beta N - \nu}{\nu+\gamma}\right) - \gamma < 0$

$\Delta = (\nu+\gamma)\left(\frac{\beta N - \nu}{\nu+\gamma}\right) > 0$

$\Rightarrow$ stable coexist exists

So, in conclusion, we have 2 major cases:

(1) if $N < \nu/\beta$

we only have 1 biologically relevant equilibrium $(N, 0)$, which is stable.

(2) if $N > \nu/\beta$

we have 2 biologically relevant equilibria,

$(N, 0)$ which is a saddle

$\left(\frac{\nu}{\beta}, \frac{N-(\nu/\beta)}{\nu+\gamma}\right)$ which is stable.

further, depending on parameter values, it can be a node or oscillatory.

pplane.

ex w oscillations

$\beta = 3, \gamma = 0.1, \nu = 4$

w/out oscillation

$\beta = 3, \gamma = 1, \nu = 4$

Note $\frac{N\beta}{\nu} = R_0$

represents the number of secondary infections generated from 1 infectious individual

R_0 = (transmission rate) · (duration of infectious period) · (# of contacts)

$= \beta \cdot \frac{1}{\nu} \cdot N$

How would you change eqns for chicken pox? spread though a school?

- Permanent immunity

$\frac{dS}{dt} = -\beta SI$

$\frac{dI}{dt} = \beta SI - \nu I$

$\frac{dR}{dt} = \nu I$

How would you change eqns for plague spread through an isolated city. over the course of a few weeks?

- Assume no births/deaths
- death due to plague.

$$\frac{dS}{dt} = -\beta SI + \gamma R$$

$$\frac{dI}{dt} = \beta SI - \nu I - \mu I$$ note μ represents death due to plague.

$$\frac{dR}{dt} = \nu I - \gamma R$$

How to model spread of black death through Europe in Medieval times
- population changes (grows) Assume only Susceptibles can reproduce
- death due to other causes

$$\frac{dS}{dt} = bS - \beta SI + \gamma R - dS$$

$$\frac{dI}{dt} = \beta SI - \nu I - \mu I - dI$$ d is death rate due to other causes.

$$\frac{dR}{dt} = \nu I - \gamma R - dR$$

How to model an infection with a long latent period, such as measles?
idea: incorporate an exposed, but not yet infectious, class

Assume a school with no population change

Facts about measles
- latency period of approx 10 - 12 days
- death due to disease
- lifetime immunity.

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dE}{dt} = \beta SI - cE$$ note c = rate at which exposed become infectious = $\frac{1}{11}$ days.

$$\frac{dI}{dt} = cE - \nu I - \mu I$$

$$\frac{dR}{dt} = \nu I$$

What about vaccinations?
Model ~~measles~~ whooping cough spread across the USA, but with vaccinations. Assume no latency period/exposure class.

$$\frac{dS}{dt} = \mu N(1-p) - dS - \beta SI$$

$$\frac{dI}{dt} = \beta SI - \nu I - dI - \mu I$$

$$\frac{dR}{dt} = \nu I$$

$$\frac{dV}{dt} = \mu Np - dI$$

p = rate at which newborns are vaccinated $\in (0,1)$

We can use this model to determine necessary value of p in order to prevent epidemics. Hint: it's not 100% herd immunity.

What about diseases for which a recovered individual remains a carrier?

Consider tuberculosis, for which some portion of individuals remain a carrier of the disease for an extended period of time

Assumptions

- Both infectious & carriers can transmit the disease, possibly at different rate
- Carriers remain in the carrier state on average $\frac{1}{\tau}$ days
- No population change
- Life long immunity

$$\frac{dS}{dt} = -\beta SI - \varepsilon\beta SC$$

$$\frac{dI}{dt} = \beta SI + \varepsilon\beta SC - \gamma I$$

$$\frac{dR}{dt} = (1-p)\gamma I + \tau C$$

$$\frac{dC}{dt} = p\gamma I - \tau C$$

where ε is a scaling factor

p is proportion of infectives, that eventually become carriers

We can also model quarantines, treatments, maternally-derived immunity, and so forth.